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Singularity of the specific heat of two-dimensional random Ising models*

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Abstract. The singularity of the specific heat is studied for the dilution ($J > J' > 0$) type and Gaussian type random Ising models using the Pfaffian method numerically. The type of singularity at the paramagnetic-ferromagnetic phase boundary is studied using the standard regression method using data up to 600×601 system size. It is shown that the logarithmic type singularity is more reliable than the double-logarithmic type and cusp type singularities. The critical temperatures are estimated accurately for both the dilution type and Gaussian type random Ising models. A phase diagram relating strength of the randomness and temperature is also presented.

PACS. 05.70.Jk Critical point phenomena – 64.60.Fr Equilibrium properties near critical points, critical exponents – 75.10.Nr Spin-glass and other random models

1 Introduction

Does weak randomness affect the universality of the two-dimensional Ising model? The pure (none-random) Ising model was solved by Onsager [1] and its universality is characterized by: 1) logarithmic divergence of the specific heat, and 2) critical exponents $\alpha = 0$, $\beta = 1/8$, $\nu = 1$, *etc.* Are these characteristics affected by randomness?

We investigate the singularity of the specific heat of two-dimensional random Ising models at the phase boundary between the paramagnetic and ferromagnetic phases. We have already reported on this problem for the $\pm J$ random Ising model [2,3]. In this paper we consider the dilution type and Gaussian type of randomness. The dilution model is defined by two different randomly distributed interactions J and J' ($J > J' > 0$). The Gaussian model is defined by random ferromagnetic-biased interactions J_{ij}.

McCoy and Wu [4] introduced a one-dimensionally-random Ising model which has translational invariance in the horizontal direction but no invariance in the vertical direction. Zittartz, Hoever and Wolff [5] extended this model and concluded by the non-perturbative method that the specific heat is finite. Therefore, this one-dimensionally-random Ising model belongs to a different universality class from the pure Ising model.

Dotsenko and Dotsenko [6] studied two-dimensionally-random Ising models. They predicted that the specific heat C diverges double-logarithmically, $C \sim \log(\log(|T - T_c|/T_c))$. Shankar *et al.* [7,8] and Shalaev [9] also studied this model and found the same behavior of the specific heat. They also showed that other physical quantities deviate logarithmically from the pure model. Blackman and Poulter [10] and Wolff and Zittartz [11] used a concentration expansion method and obtained results that support the double-logarithmic singularity of the specific heat. On the other hand, cusp type (non-divergent) behavior of the specific heat was predicted by Tamaribuchi and Takano [12] and Harris [13].

To confirm theoretical predictions, Wang *et al.* [14] and Wiseman *et al.* [15] performed Monte-Carlo simulations for the dilution type random Ising model. Their results support the double-logarithmic divergence of the specific heat. They also found that the exponents for other physical quantities are the same as those of the pure Ising model. Stauffer *et al.* [16] applied the transfer-matrix method to the finite width(≤ 18) long strips and supported the double-logarithmic behavior. Kim *et al.* [17] claimed that the exponent ν is dependent on the strength of randomness. Their result has not yet been supported [18].

Kitatani and Oguchi [19] showed that $\nu = 1$ using the transfer-matrix method at the paramagnetic-ferromagnetic-spinglass tri-critical point. Ito, Matsuhisa and Kitatani [20] performed the Monte-Carlo simulations for the $\pm J$ model at the weakly random region and obtained the critical exponent $\beta = 0.13 \pm 0.01$.

This paper is organized as follows. The method and samples we used in this study are explained in Section 2. Section 3 is devoted to the dilution model. Here we compare our data with the result by Wang *et al.* [14]. The Gaussian model is discussed in Section 4. The discussion and conclusions are given in Section 5.

* Dedicated to J. Zittartz on the occasion of his 60th birthday

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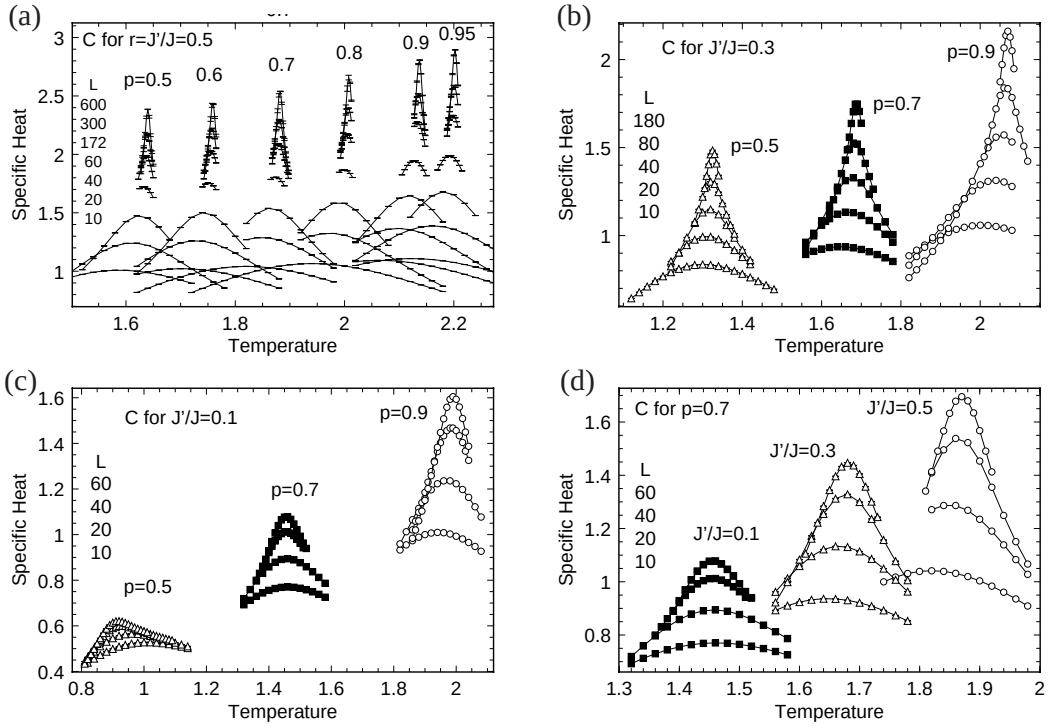

Fig. 1. Specific heat for the dilution model. (a) $J'/J = 0.5$ for $p = 0.95, 0.9, 0.8, 0.7, 0.6, 0.5$, (b) $J'/J = 0.3$ for $p = 0.9, 0.7, 0.5$, (c) $J'/J = 0.1$ for $p = 0.9, 0.7, 0.5$. (d) The case $p = 0.7$ is provided here to show the J'/J-dependence on the same scale. Points are interpolated for guidance. Errors are smaller than the symbols.

2 Method and samples

2.1 Pfaffian method

For a two-dimensional Ising model, the exact free energy can be obtained by the Pfaffian method [3,4,10,11,21–23]. The Hamiltonian is defined by $\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} S_i S_j$ with $S_i = \pm 1$. This method represents the partition function as a determinant of an antisymmetric real $4N \times 4N$ matrix D, *i.e.*,

$$Z = 2^N \prod_{\langle ij \rangle} \cosh(J_{ij}/kT) |D|^{1/2}, \tag{1}$$

where N is the number of spins and D has elements $0, \pm 1$ and $\tanh(J_{ij}/kT)$. We compute directly and numerically the determinant for each sample at each temperature. The specific heat C is obtained as a numerical derivative of $\log Z$. Technical details are given in references [2,3]. This method gives the numerically exact free energy. We do not encounter the equilibrium problem which occurs in Monte-Carlo simulations.

2.2 Samples

We let a horizontal size be L with a periodic boundary condition and let a vertical size be $L+1$ with a free boundary condition. The number of spins is $N = L \times (L+1)$.

For the dilution model, the probability distribution $P(J_{ij})$ is defined by

$$P(J_{ij}) = p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} - J'), \tag{2}$$

where we choose $J = 1$ and $J' = 0.5, 0.3, 0.1$.

The true free energy of a quenched system is approximated by sample average

$$\mathcal{F} = \int P(J_{ij}) F(J_{ij}) dJ_{ij} \simeq \frac{1}{number\,of\,samples} \sum_{sample} F(J_{ij}). \tag{3}$$

We introduce a modified probability distribution $P'(J_{ij})$ defined by [24]

$$P'(J_{ij}) = P(J_{ij}) \times \delta(p - x), \tag{4}$$

where

$$x = \frac{number\,of\,J\,interactions}{number\,of\,all\,interactions}\,. \tag{5}$$

The free energy of the dilution model is computed by equation (3) with $P'(J_{ij})$ instead of $P(J_{ij})$.

For the Gaussian model the probability distribution $P(J_{ij})$ is defined by

$$P(J_{ij}) = \frac{1}{\sqrt{\pi\sigma}} \exp(-\frac{(J_{ij} - J_m)^2}{\sigma}), \tag{6}$$

and its free energy is given by equation (3) as it is. The mean value of the interactions is chosen as $J_m = 1$.

We take 20 000 $\sim$ 2 sample averages for L= 5 $\sim$ 600 lattices. We assume large lattices do not require many samples. The typical variance (one σ) of the specific heat is 1.2×10^{-4} for $L = 5$ and 1.7×10^{-3} for $L = 600$. These are shown in Figure 1a.

3 Dilution model

3.1 Specific heat

The specific heat C is presented in Figure 1 for three cases, $J'/J = 0.5$, $J'/J = 0.3$ and $J'/J = 0.1$. We observe that the specific heat increases and its peak sharpens as p increases and as J'/J increases.

We find the maximum value of the specific heat by extrapolating the three largest values around each peak with a quadratic function of the temperature. The maximum value $C_{max}(L)$ and the critical temperature $T_c(L)$ are obtained by solving this function.

To examine the singularity of the specific heat, we plot its maximum value $C_{max}(L)$ *versus* the system size L in Figure 2. We test the three types of regression equation with three fitting parameters a, b and c by the least squares method.

The logarithmic type regression equation is defined by

$$C^l_{max}(L) = a + b \log(L + c). \tag{7}$$

This is the same as the pure system.

The double-logarithmic type regression equation is

$$C^d_{max}(L) = a + b \log(\log L + c), \tag{8}$$

which is predicted by Dotsenko *et al.* [6], Shalaev [9] and Shankar *et al.* [7].

The cusp type regression equation is

$$C^c_{max}(L) = a + b \log L/(\log L + c), \tag{9}$$

which does not diverge [12,13].

In all regression equations, the major finite size correction will be included in the parameter c.

The results of the regressions are summarized in Tables 1–4 and Figure 2. The value r in the last column of the tables is a so-called coefficient of determination [25] defined by

$$r \equiv \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2 \sum_i (y_i - \bar{y})^2}}, \tag{10}$$

where x_i is a variable (here it is L) and y_i is $C_{max}(L)$, and $\bar{x}$ and $\bar{y}$ are averages. When $r = 1$, the regression is perfect and the data are completely fitted to the regression equation.

We cannot distinguish the three different regression curves in Figure 2 at this scale. The value r is almost 1 for every type and every case. Even for the pure case (see Tab. 1), which is known to have logarithmic singularity, three types of regression at the present range of sizes L = 5 $\sim$ 700 give no definite conclusion.

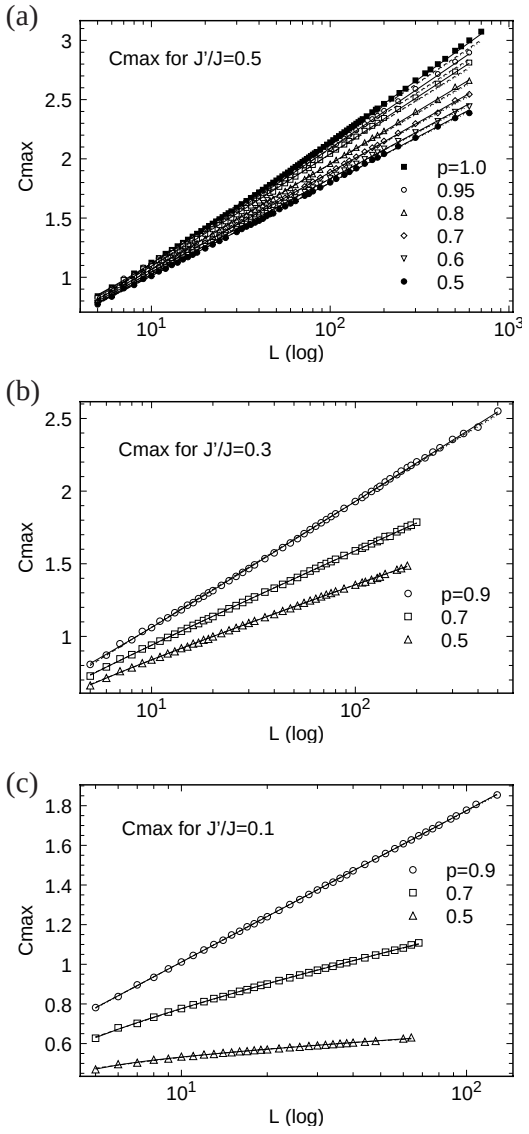

Fig. 2. The maximum value C_{max} *versus* $\log L$. (a) J'/J = 0.5, (b) $J'/J = 0.3$, and (c) $J'/J = 0.1$. Solid, broken and dotted lines are logarithmic, double-logarithmic and cusp type regression results, respectively.

In Tables 1–4 we see that the parameter c for the double-logarithmic type (Eq. (8)) and the cusp type (Eq. (9)) of regression is large enough to expand by $\log L/c$ as follows [2]:

$$C^d_{max}(L) \simeq a + b \log c + (b/c) \log L + \cdots, \tag{11}$$

$$C^c_{max}(L) \simeq a + (b/c) \log L - (b/c^2)(\log L)^2 + \cdots \tag{12}$$

This means that for large c these two types of regression equation can be regarded as the logarithmic type equation

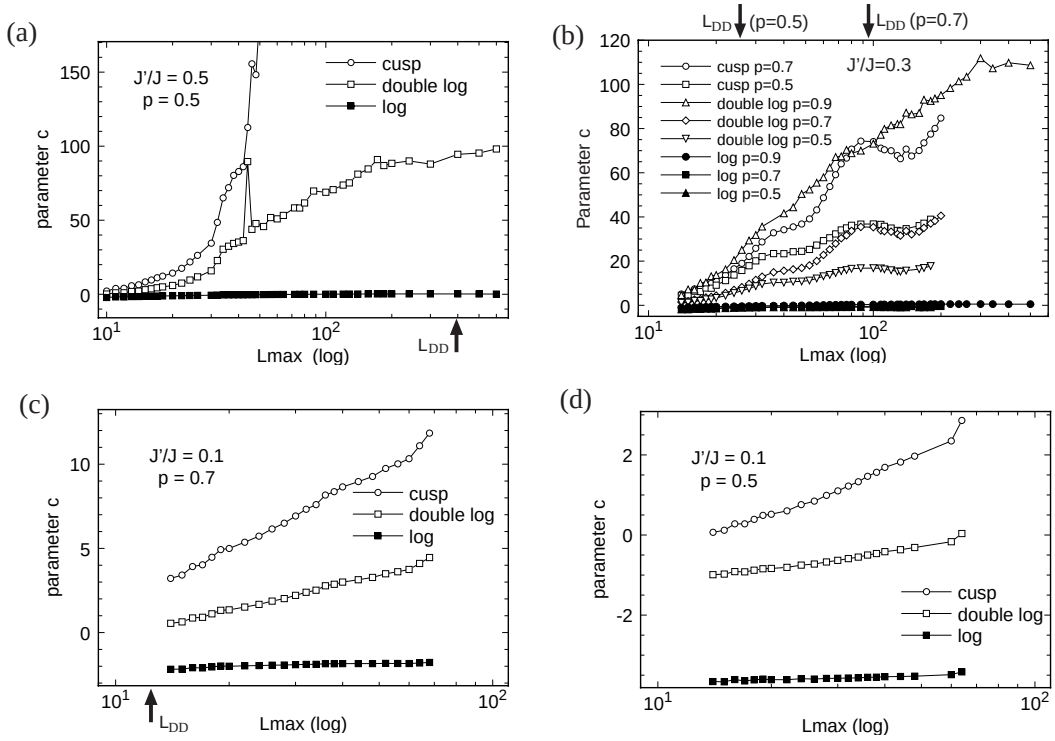

Fig. 3. The parameter c in three different types of regression equation *versus* $\log L_{max}$. The crossover size L_{DD} is indicated by arrows.

and that logarithmic data can be fitted using the double-logarithmic and/or cusp type regression equations with a large parameter c in a finite range of the regression region. This is the reason data for the pure model fit very well to any equation.

Let the regression range be $5 \leq L \leq L_{max}$. The behavior of the parameter c is examined by varying L_{max} in Figure 3. We used the SIMPLEX nonlinear fitting method. Small irregularities in C_{max} yield big errors in c. Ignoring such exceptional irregularities, we find that c for double-logarithmic and cusp type regression equations increases almost monotonically as L_{max} increases. On the other hand, c is nearly constant for the logarithmic regression equation. We expect, then, that c of C^d_{max} and C^c_{max} will be infinite in the thermodynamic limit ($L_{max} \to \infty$).

Thus, we conclude that the specific heat for the dilution model diverges logarithmically.

To examine our conclusion further, we consider the following.

Dotsenko *et al.* [6] reported on the crossover system size L_{DD}. The specific heat seems logarithmic when the system size is smaller than L_{DD} but it seems double-logarithmic when the system size is larger. The crossover size L_{DD} is given by [6,14]

$$\begin{aligned} L_{DD} &= \exp(\pi/4g_0(p,J',J)), \\ g_0(p,J',J) &= (1-p)a^2/(1+ab)^2, \\ a &= (\tanh(\beta_c J') - \tanh(\beta_c J))/\tanh(\beta_c J), \\ b &= \tanh(\beta_c J)/2\sqrt{2}, \end{aligned} \tag{13}$$

where β_c is the inverse of the critical temperature for the pure Ising model satisfying $\tanh(\beta_c J) = \sqrt{2} - 1$. The values of L_{DD} for the systems we studied are given in Table 5 and indicated by arrows in Figure 3. If the argument of Dotsenko *et al.* is correct, our lattice size is large enough to detect the crossover and the behavior of c may change at this crossover size, *i.e.*, c may become constant above L_{DD}. However, we do not observe such in Figure 3.

Let us consider the results of Wang *et al.* [14]. Their extensive Monte-Carlo simulations clearly show double-logarithmic behavior of the specific heat. By rescaling and fitting their data to equation (8), we obtain the regressed curves:

$$\begin{aligned} &-0.991+1.19\log(2.274+\log L) \ \text{ for } J'/J=0.25 \text{ and } p=0.5, \\ &0.161+0.309\log(0.4911+\log L) \ \text{ for } J'/J=0.1 \text{ and } p=0.5. \end{aligned} \tag{14}$$

Here we restrict their data up to $L = 181$ (almost equal to our largest system) to compare our data on equal condition. The crossover size ($L_{DD} \simeq \exp(c)$) is then estimated as $L_{DD} \simeq 9.7$ for $J'/J = 0.25$ and $L_{DD} \simeq 1.6$ for $J'/J = 0.1$. These values agree with the theoretical values of 10.1 for $J'/J = 0.25$ and 4.4 for $J'/J = 0.1$ [6,14]. This result does not change when we use their whole data (up to $L = 600$).

Our present data, however, yield a much larger crossover size. From the data in Tables 2–4, the estimated values are $L_{DD} \simeq 4.1 \times 10^{42}$, 5.3×10^7 and 1.04 for $J'/J = 0.5, 0.3$ and 0.1 ($p = 0.5$), respectively. Such

Table 1. The constants in the three types of regression equations (7),(8) and (9) for the maximum value $C_{max}(L)$ of the specific heat in the case of the pure Ising model. Size denotes the maximum size used for regression, error signifies a confidence interval of 95% , and r is a coefficient of determination. T_c row is the critical temperature calculated from equation (17).

regression	size	a	(error)	b	(error)	c	r
log	700	−4.115e−02	(4.6e−03)	4.723e−01	(1.2e−04)	1.613e+00	0.9999503
double-log	700	−3.011e+02	(3.1e+00)	6.143e+01	(6.7e−02)	1.347e+02	0.9993220
cusp	700	9.494e−02	(1.7e−02)	6.472e+15	(6.1e+12)	1.457e+16	0.9994902
T_c	700	2.269e+00	(2.2e−05)	−9.230e−01	(5.7e−05)	1.268e+00	0.9999965

Table 2. The constants in the regression equations for C_{max} in the case of $J'/J = 0.5$.

p	size	a	(error)	b	(error)	c	r
logarithmic							
0.95	600	2.230e−02	(7.9e−03)	4.477e−01	(2.8e−04)	1.293e+00	0.9998893
0.90	600	6.208e−02	(6.4e−03)	4.289e−01	(2.3e−04)	1.080e+00	0.9999180
0.80	600	1.253e−01	(5.8e−03)	3.972e−01	(2.0e−04)	7.039e−01	0.9999198
0.70	600	1.710e−01	(5.3e−03)	3.722e−01	(1.9e−04)	4.295e−01	0.9999227
0.60	600	2.030e−01	(5.6e−03)	3.536e−01	(2.0e−04)	2.142e−01	0.9999066
0.50	600	2.168e−01	(6.0e−03)	3.433e−01	(2.1e−04)	1.307e−01	0.9998859
double-logarithmic							
0.95	600	−2.848e+02	(3.7e+00)	5.816e+01	(1.1e−01)	1.340e+02	0.9993209
0.90	600	−2.700e+02	(3.0e+00)	5.534e+01	(8.5e−02)	1.318e+02	0.9995068
0.80	600	−2.179e+02	(1.8e+00)	4.589e+01	(5.4e−02)	1.157e+02	0.9996873
0.70	600	−1.881e+02	(1.3e+00)	4.029e+01	(3.7e−02)	1.069e+02	0.9997995
0.60	600	−1.658e+02	(8.6e−01)	3.608e+01	(2.6e−02)	9.951e+01	0.9998671
0.50	600	−1.588e+02	(8.2e−01)	3.468e+01	(2.5e−02)	9.812e+01	0.9998618
cusp							
0.95	600	1.365e−01	(1.9e−02)	4.730e+05	(7.5e+02)	1.118e+06	0.9994779
0.90	600	1.544e−01	(1.5e−02)	2.269e+09	(3.0e+06)	5.547e+09	0.9996372
0.80	600	1.820e−01	(1.0e−02)	3.012e+15	(2.9e+12)	7.825e+15	0.9997912
0.70	600	2.039e−01	(7.3e−03)	2.716e+09	(1.9e+06)	7.440e+09	0.9998728
0.60	600	2.188e−01	(5.9e−03)	4.644e+14	(2.8e+11)	1.326e+15	0.9998983
0.50	600	2.262e−01	(6.1e−03)	2.185e+15	(1.4e+12)	6.403e+15	0.9998828

Table 3. The constants in the regression equations for C_{max} in the case of $J'/J = 0.3$.

p	size	a	(error)	b	(error)	c	r
logarithmic							
0.90	500	1.574e−01	(5.4e−03)	3.843e−01	(1.8e−04)	5.155e−01	0.9999239
0.70	200	3.189e−01	(3.4e−03)	2.761e−01	(1.3e−04)	−5.205e−01	0.9999399
0.50	180	3.628e−01	(3.2e−03)	2.160e−01	(1.3e−04)	−9.300e−01	0.9999124
double-logarithmic							
0.90	500	−1.975e+02	(1.2e+00)	4.217e+01	(3.5e−02)	1.086e+02	0.9998029
0.70	200	−4.580e+01	(1.6e−01)	1.244e+01	(6.3e−03)	4.055e+01	0.9999309
0.50	180	−1.350e+01	(6.3e−02)	4.778e+00	(3.1e−03)	1.779e+01	0.9999018
cusp							
0.90	500	1.970e−01	(7.5e−03)	6.059e+10	(4.0e+07)	1.612e+11	0.9998681
0.70	200	2.522e−01	(3.9e−03)	2.594e+01	(1.3e−02)	8.472e+01	0.9999308
0.50	180	2.574e−01	(3.9e−03)	1.039e+01	(6.7e−03)	3.896e+01	0.9999012

Table 4. The constants in the regression equations for C_{max} in the case of $J'/J = 0.1$.

p	size	a	(error)	b	(error)	c	r
logarithmic							
0.90	128	2.457e−01	(2.4e−03)	3.322e−01	(1.1e−04)	7.335e−03	0.9999801
0.70	68	4.465e−01	(2.8e−03)	1.568e−01	(1.6e−04)	−1.775e+00	0.9998707
0.50	64	4.519e−01	(1.8e−03)	4.256e−02	(1.2e−04)	−3.418e+00	0.9992529
double-logarithmic							
0.90	128	−9.871e+01	(3.3e−01)	2.348e+01	(1.2e−02)	6.763e+01	0.9999551
0.70	68	−1.721e+00	(2.2e−02)	1.306e+00	(1.9e−03)	4.457e+00	0.9997440
0.50	64	3.951e−01	(4.1e−03)	1.597e−01	(6.8e−04)	3.673e−02	0.9982141
cusp							
0.90	128	2.463e−01	(2.4e−03)	2.292e+14	(7.3e+10)	6.902e+14	0.9999800
0.70	68	2.436e−01	(5.7e−03)	3.265e+00	(4.8e−03)	1.185e+01	0.9997328
0.50	64	2.459e−01	(8.0e−03)	6.382e−01	(2.9e−03)	2.860e+00	0.9979654

Table 5. Crossover size L_{DD} calculated from equation (13).

J'/J	$p = 0.95$	$p = 0.9$	$p = 0.8$	$p = 0.7$	$p = 0.6$	$p = 0.5$
0.5	1.0×10^{25}	1.0×10^{12}	3.2×10^{6}	2.2×10^{4}	1.8×10^{3}	398
0.3		8.5×10^{5}		95		15
0.1		1683		12		4.4

large values allow expansion by $\log L/c$ as given in equation (11), *i.e.*, our data are suitable for the logarithmic behavior. The case of $J'/J = 0.1$ and $p = 0.5$ is not clear. The value of c is reasonably small when assuming double-logarithmic behavior of the specific heat. However, c increases monotonically in the present regression region (see Fig. 3d) and is expected to become much larger than the present value.

Finally, it is known that $p = 0.5$ is a special point where the system is self-dual and that the exact critical temperature T_c^{exact} is given by [26,27]

$$\sinh(2\beta_c J)\sinh(2\beta_c J') = 1, \qquad \beta_c = 1/T_c^{exact}. \quad (15)$$

Thus, we have the following T_c^{exact} values for the present systems:

$$T_c^{exact} = \begin{cases} 1.64101793 & \text{for } J'/J = 0.5 \\ 1.32884150 & \text{for } J'/J = 0.3 \\ 0.905883062 & \text{for } J'/J = 0.1. \end{cases} \quad (16)$$

By extrapolating the raw data we estimate the value of the specific heat $C(T_c^{exact})$ at these temperatures. Of course, $C(T_c^{exact})$ is always smaller than C_{max}.

Applying the same analysis to the $C(T_c^{exact})$ values, we obtain the same results as for C_{max}. The three types of regression are fitted almost perfect and L_{max}-dependence of the parameter c has the similar tendency as for the C_{max} data.

Thus, we conclude that the specific heat of the dilution type random Ising model diverges logarithmically as does the pure Ising model.

3.2 Critical temperature T_c

We assume the critical exponent $\nu = 1$. The regression equation is then defined by

$$T_c(L) = a + \frac{b}{L + c}\cdot \quad (17)$$

The parameter a is the critical temperature in the thermodynamic limit. Results are summarized in Table 6 and Figure 4. We obtain a very good fit and so assume that the regression equation is valid. The values a for $p = 0.5$ agree very well with the exact values given by equation (16). The parameter b changes sign depending on p as in the $\pm J$ model [2].

The critical temperatures T_c (parameter a in Eq. (17)) in the thermodynamic limit are the p-dependent as shown in Figure 5. Interpolation is carried out by

$$a(p) = a_0 + a_1(1-p) + a_2(1-p)^2, \quad (18)$$

where the fitting parameters a_0, a_1 and a_2 are as given in Table 7. This line denotes the phase boundary in the $p-T$ phase diagram. The paramagnetic phase is above the line and the ferromagnetic phase is below it. The line seems to be concave ($a_2 > 0$) for the dilution model whereas for the Gaussian model and the $\pm J$ model the lines are convex ($a_2 < 0$).

4 Gaussian model

4.1 Specific heat

A similar argument as for the dilution model is applied to the Gaussian random model. We choose the mean value

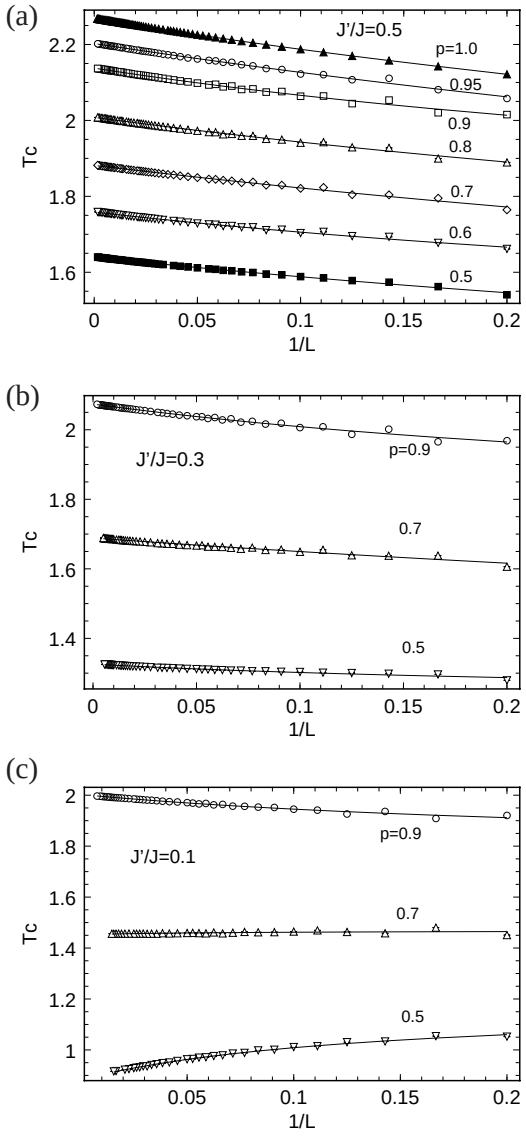

Fig. 4. Critical temperature $T_c(L)$ *versus* the inverse of the system size $1/L$. The lines are given by equation (17).

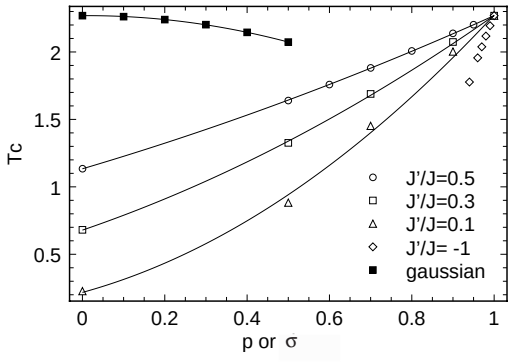

Fig. 5. The $p(\sigma)$-dependence of the critical temperature for the dilution (Gaussian) model. Curves are given by equation (18) and the data in Table 7. We have added the data of the $\pm J$ model [2].

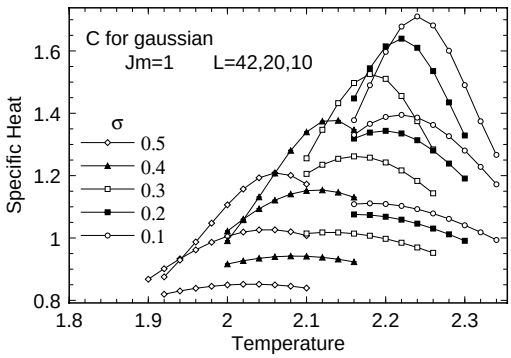

Fig. 6. The specific heat for the Gaussian model for $\sigma = 0.1$ – 0.5. $J_m = 1$ for all cases.

of interactions in equation (6) as $J_m = 1$ for all cases. We vary the variance as $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5$. For these parameters, this Gaussian model still has the paramagnetic-ferromagnetic phase transition as shown by McMillan [28]. The spinglass phase exists at $T = 0$ and for $\sigma/J_m \geq 0.961$, and it is far from our region.

Since some interactions J_{ij} may become negative, this Gaussian model does not free from the frustrations [29] which are completely inhibited by the dilution model. Frustrations are large for large σ.

The specific heat is given in Figure 6, and the maximum value $C_{max}(L)$ are plotted *versus* $\log L$ in Figure 7.

Similarly as in the previous section, we have examined the three regression equations (7), (8) and (9). The coefficients of determination r vary only slightly between the regressions (Tab. 8). Thus, we cannot determine which equation gives the best-fitting curve and so we turn our attention to the L_{max}-dependence of c (Fig. 8).

We observe that c almost monotonically increases as a function L_{max} for the double-logarithmic and cusp type regression equations. The value of c is also large. Thus, we conclude as in the dilution case that the specific heat diverges logarithmically.

4.2 Critical temperature T_c

We use the same regression equation (17) to analyse the size L-dependence of the critical temperature for each σ value. The results are given in Table 9 and Figure 9.

The p-dependence of the critical temperature a in the thermodynamic limit is given in Table 7 and Figure 5. Note that the case $\sigma = 0$ is a pure Ising model.

5 Discussion and conclusions

Two types of random Ising model are investigated using the Pfaffian method. We have studied the size-dependence of the maximum value of the specific heat using three types of regression equation. Analysing the

Table 6. T_c for $J'/J = 0.5$, 0.3 and 0.1.

p	size	a	(error)	b	(error)	c	r
$J'/J = 0.5$							
0.95	600	2.202e+00	(9.0e−04)	−8.269e−01	(2.3e−03)	8.883e−01	0.9975594
0.90	600	2.138e+00	(1.1e−03)	−8.470e−01	(3.1e−03)	1.822e+00	0.9958298
0.80	600	2.008e+00	(8.0e−04)	−7.283e−01	(2.1e−03)	1.176e+00	0.9973497
0.70	600	1.881e+00	(9.0e−04)	−6.491e−01	(2.3e−03)	9.259e−01	0.9960805
0.60	600	1.759e+00	(6.9e−04)	−6.209e−01	(1.9e−03)	1.702e+00	0.9971050
0.50	600	1.640e+00	(5.3e−04)	−5.699e−01	(1.4e−03)	1.094e+00	0.9981990
$J'/J = 0.3$							
0.90	500	2.074e+00	(1.2e−03)	−8.032e−01	(3.6e−03)	2.357e+00	0.9926086
0.70	200	1.689e+00	(1.2e−03)	−4.221e−01	(3.0e−03)	8.219e−01	0.9854384
0.50	180	1.326e+00	(7.9e−04)	−3.159e−01	(2.5e−03)	3.155e+00	0.9845816
$J'/J = 0.1$							
0.90	128	2.003e+00	(1.7e−03)	−7.841e−01	(5.7e−03)	3.598e+00	0.9899089
0.70	68	1.452e+00	(4.1e−03)	2.366e−01	(2.3e−02)	1.401e+01	0.5518048
0.50	64	8.828e−01	(2.4e−03)	2.175e+00	(1.0e−02)	7.279e+00	0.9978952

Table 7. The $p(\sigma)$-dependence of the critical temperature T_c for the dilution (Gaussian) model. The parameters a_0, a_1 and a_2 are defined by equation (18).

J'/J	a_0	(error)	a_1	(error)	a_2	(error)	r
0.5	2.2696	(3.0e−04)	−1.369	(1.2e−02)	2.333	(1.3e−02)	0.99998866
0.3	2.2702	(1.0e−04)	−2.135	(4.4e−02)	5.440	(4.9e−02)	0.99993145
0.1	2.2734	(4.7e−04)	−3.271	(2.0e−01)	1.216	(2.2e−01)	0.99924194
Gaussian	2.2691	(8.7e−05)	2.064	(6.4e−03)	−0.824	(1.5e−02)	0.99996720

Table 8. The constants in the regression equations for C_{max} for the Gaussian model.

σ	size	a	(error)	b	(error)	c	r
logarithmic							
0.1	600	−9.101e−03	(7.9e−03)	4.574e−01	(2.5e−04)	1.464e+00	0.9998870
0.2	600	7.495e−02	(7.0e−03)	4.177e−01	(2.0e-04)	1.002e+00	0.9998833
0.3	88	1.943e−01	(4.1e−03)	3.588e−01	(1.9e−04)	2.131e−01	0.9999494
0.4	46	2.784e−01	(8.2e−04)	2.988e−01	(5.4e−05)	−2.292e−01	0.9999974
0.5	68	3.292e−01	(4.1e−03)	2.414e−01	(2.3e−04)	−4.324e−01	0.9998884
double-logarithmic							
0.1	600	−3.008e+02	(3.9e+00)	6.105e+01	(1.0e−01)	1.381e+02	0.9992812
0.2	600	−2.502e+02	(2.4e+00)	5.180e+01	(6.1e−02)	1.256e+02	0.9995144
0.3	88	−1.086e+02	(5.7e−01)	2.566e+01	(2.1e−02)	6.945e+01	0.9998896
0.4	46	−5.500e+01	(6.6e−02)	1.451e+01	(3.3e−03)	4.502e+01	0.9999960
0.5	68	−3.414e+01	(2.1e−01)	9.617e+00	(9.8e−03)	3.581e+01	0.9998717
cusp							
0.1	600	1.102e−01	(1.9e−02)	1.351e+20	(1.8e+17)	3.120e+20	0.9994411
0.2	600	1.570e−01	(1.4e−02)	1.711e+26	(1.7e+23)	4.274e+26	0.9996493
0.3	88	2.132e−01	(4.6e−03)	3.801e+15	(2.4e+12)	1.073e+16	0.9999348
0.4	46	2.392e−01	(9.5e−04)	3.510e+01	(7.0e−03)	1.100e+02	0.9999968
0.5	68	2.748e−01	(4.7e−03)	2.008e+01	(2.1e−02)	7.489e+01	0.9998715

Table 9. T_c for the Gaussian model.

σ	size	a	(error)	b	(error)	c	r
0.1	600	2.262e+00	(6.6e−05)	−9.112e−01	(1.8e−04)	1.262e+00	0.9999849
0.2	600	2.240e+00	(1.2e−04)	−8.851e−01	(3.2e−04)	1.316e+00	0.9999350
0.3	88	2.202e+00	(2.4e−04)	−8.128e−01	(6.4e−04)	1.230e+00	0.9998912
0.4	46	2.145e+00	(1.3e−03)	−5.820e−01	(3.0e−03)	−3.745e−02	0.9979210
0.5	68	2.073e+00	(1.9e−03)	−3.948e−01	(3.8e−03)	−1.028e+00	0.9895097

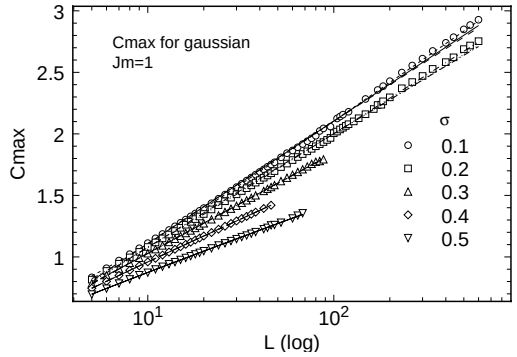

Fig. 7. The maximum value of the specific heat for the Gaussian model. Solid, broken and dotted lines are logarithmic, double-logarithmic and cusp type regression results, respectively.

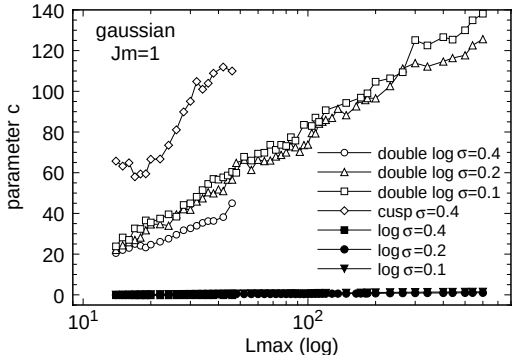

Fig. 8. The parameter c in regression equations *versus* $\log L_{max}$ for the Gaussian model.

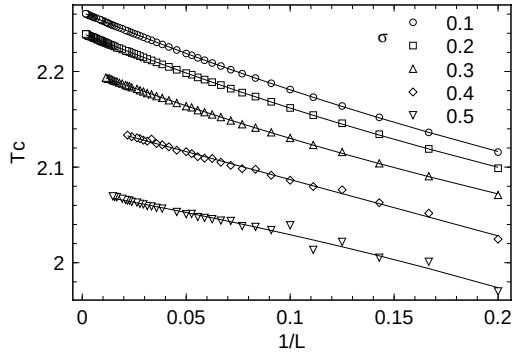

Fig. 9. Critical temperature *versus* $1/L$ for the Gaussian model. The lines are given by equation (17).

L_{max}-dependence of the fitting parameter c in the regression equations (7),(8) and (9), we conclude that the specific heat of both systems diverges logarithmically. The universality of the weakly-random Ising model is strongly conserved with respect not only to the value of the exponent but also its functional form.

We have previously reported that the specific heat of the $\pm J$ model diverges logarithmically [2]. We compare the maximum value of the specific heat of the dilution model (Fig. 2) and that of the $\pm J$ model (Fig. 3 in Ref. [2]) for $L \leq 600$. The former value is always larger than the latter value. If we accept the logarithmic singularity of the $\pm J$ model, the reasonable conclusion is that the present dilution model also has the same logarithmic singularity.

Why are our results and those of Wang *et al.* [14] different? Differences may arise from 1) the shape of lattice and the boundary condition, and 2) the method used to evaluate the specific heat. Wang *et al.* used the Monte-Carlo method whereas we used the Pfaffian method. Differences due to the first condition are expected to disappear for large lattices. Regarding methods, both have weak points. The Monte-Carlo method has an equilibrium problem. Since the specific heat is the energy fluctuation, *i.e.*, $C = (\langle E^2 \rangle - \langle E \rangle^2)/T^2$, simulations should be carried out to include important(large) fluctuations. The Pfaffian method cannot give the specific heat directly. We should differentiate the free energy twice numerically. The number of samples is also limited in both methods. We cannot take every interaction configuration for large systems, therefore, we approximate the physical quantity by sample average. In order to gain a better understanding of the differences between our results and Wang *et al.*'s, we need larger size, more samples and longer computations.

Critical temperatures are estimated accurately for the dilution type and Gaussian type models. The phase boundary curve is concave for the dilution model, but it is convex for the Gaussian and $\pm J$ models [2]. The Gaussian and $\pm J$ models have frustrations and a spinglass phase [30–34], whereas the dilution model does not.

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